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q-deformed supersymmetric Newton oscillators

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Abstract. We construct a two-parameter deformed superoscillator algebra of n-bosonic and m-fermionic q-deformed Newton oscillators. These deformed oscillators are invariant under the undeformed Lie supergroup $U(n \mid m)$. It is shown that the bilinears constructed from the annihilation and creation operators of this deformed superoscillator algebra satisfy (q_1, q_2)-deformed commutation relations which can be used to define a deformed Lie superalgebra $osp_{q_1,q_2}(2n \mid 2m, R)$ whose Lie super subalgebra $u(n \mid m)$ is undeformed.

1 Introduction

q-deformations of Lie groups and Lie algebras have yielded many applications in a wide spectrum of research covering mathematical physics, statistical physics and high energy physics. Due to the fact that these q-deformations can be considered as quantum groups [1,2], another interesting research field has spontaneously arisen, concerned with the relation between quantum groups, their associated algebras and q-deformed oscillator systems. Meanwhile oscillator constructions of several quantum groups and algebras have extensively been investigated [3–5]. Recently, another interesting paper related to this field, starting from the n-harmonic oscillator realization of the Lie algebra $sp(2n, R)$, has shown that this approach is useful in the study of integrable systems [6]. In this context, it is natural that generalizations of these simple Lie algebras and Lie groups to Lie superalgebras and supergroups have been considered by some researchers [7–9]. However, in the early days after the introduction of this notion, the oscillator-like constructions of some non-compact groups and supergroups were derived and subsequently applied to the unitary representations of the supergroups of extended supergravity theories [10]. In the past decade several such investigations have become strongly connected with the supersymmetric theories. Roughly speaking, a supersymmetric algebra can be considered as a supersymmetric quantum mechanics involving the fermionic and the bosonic operators and therefore it unifies the quantum statistics between Bose and Fermi statistics. It should be mentioned now that a generalized statistics was introduced by considering it to be a possible extension of quantum statistics [11]. In particular, several forms of supersymmetry structures such as parasupersymmetry or orthosupersymmetry have extensively been studied and found applications in some physical models, e.g. the quantum Hall effect [12]. The idea of supersymmetry and its development is one of the cornerstones of the construction of field and string theories [13].

The aim of this letter is to construct some particular deformed Lie algebras. We will construct these algebras, $sp_{q_1}(2n, R)$ and $so_{q_2}(2m, R)$, by considering the bosonic and fermionic multidimensional q-deformed Newton oscillators which are invariant under the undeformed Lie groups $U(n)$ and $U(m)$, respectively. We then extend our construction to describe the defining relations for the (q_1, q_2)-deformed Lie superalgebra $osp_{q_1,q_2}(2n \mid 2m, R)$ by using n-bosonic and m-fermionic q-deformed Newton oscillators invariant under the undeformed Lie supergroup $U(n \mid m)$.

2 The construction of the deformed Lie algebras $sp_{q_1}(2n, R)$ and $so_{q_2}(2m, R)$

Let us first recall that the q_1-deformed bosonic Newton [14,15] oscillator satisfies the following relations:

$$\begin{aligned} a_i a_j^+ - q_1 a_j^+ a_i &= H\delta_{ij}, \\ a_i a_j &= a_j a_i, \\ a_i H &= q_1 H a_i, \quad i, j = 1, 2, ..., n, \end{aligned} \tag{1}$$

where a^+ and a are the bosonic creation and annihilation operators, respectively. By rescaling a and a^+, H can be considered as q_1^N, where N is the total number operator and $q_1 \in R, q_1 > 0$.

We now take the n-bosonic q_1-deformed Newton oscillator invariant under the undeformed Lie group $U(n)$ which satisfies the relations given by (1). The bilinears constructed from the annihilation and creation operators of this oscillator are as follows:

$$T_{ij} = a_i^+ a_j + \frac{1}{2q_1}(H\delta_{ij}) = \frac{1}{2}\left(a_i^+ a_j + \frac{1}{q_1} a_j a_i^+\right), \tag{2}$$

$$K_{ij} = a_i a_j, \tag{3}$$

$$L_{ij} = a_i^+ a_j^+ = K_{ji}^+, \quad i,j = 1,2,...,n. \tag{4}$$

These generators define a deformed Lie algebra $sp_{q_1}(2n, R)$ and they satisfy the commutation relations

$$[T_{ij}, T_{kl}] = \frac{H}{q_1}\left(\delta_{jk}T_{il} - \delta_{li}T_{kj}\right), \tag{5}$$

$$[T_{ij}, K_{lm}]_{q_1^{-2}} = -\frac{H}{q_1}\left(\delta_{li}K_{mj} + \delta_{mi}K_{lj}\right), \tag{6}$$

$$[T_{ij}, L_{mn}]_{q_1^2} = \frac{H}{q_1}\left(\delta_{jm}L_{in} + \delta_{jn}L_{im}\right), \tag{7}$$

$$[K_{ij}, K_{lm}] = [L_{ij}, L_{lm}] = 0, \tag{8}$$

$$[K_{ij}, L_{mn}]_{q_1^4} = q_1^2 H\left(\delta_{jm}T_{ni} + \delta_{im}T_{nj} + \delta_{jn}T_{mi} + \delta_{in}T_{mj}\right), \tag{9}$$

and

$$[T_{mn}, H] = 0, \tag{10}$$

$$[K_{mn}, H]_{q_1^2} = 0, \tag{11}$$

$$[L_{mn}, H]_{q_1^{-2}} = 0, \tag{12}$$

where $[A,B]_{q_1} = AB - q_1 BA$.

On the other hand, the q_2-deformed fermionic Newton oscillator satisfies the defining commutation relations:

$$\begin{aligned} f_\mu f_\nu^+ + q_2 f_\nu^+ f_\mu &= H\delta_{\mu\nu}, \\ f_\mu f_\nu &= -f_\nu f_\mu, \\ f_\mu H &= q_2 H f_\mu, \quad \mu,\nu = 1,2,...,m, \end{aligned} \tag{13}$$

where f^+ and f are the fermionic creation and annihilation operators, respectively. By rescaling f and f^+, H can be considered as q_2^N, where N is the total number operator and $q_2 \in R, q_2 > 0$.

We take the m-fermionic q_2-deformed Newton oscillators invariant under the undeformed group $U(m)$ and choose bilinears constructed from the annihilation and creation operators of this oscillator as

$$T_{\mu\nu} = f_\mu^+ f_\nu - \frac{1}{2q_2}(H\delta_{\mu\nu}) = \frac{1}{2}\left(f_\mu^+ f_\nu - \frac{1}{q_2} f_\nu f_\mu^+\right), \tag{14}$$

$$K_{\mu\nu} = f_\mu f_\nu, \tag{15}$$

$$L_{\mu\nu} = f_\mu^+ f_\nu^+ = K_{\nu\mu}^+, \quad \mu,\nu = 1,2,...,m. \tag{16}$$

By using these generators, the following commutation relations can be verified:

$$[T_{\mu\nu}, T_{\alpha\beta}] = \frac{H}{q_2}\left(\delta_{\nu\alpha}T_{\mu\beta} - \delta_{\beta\mu}T_{\alpha\nu}\right), \tag{17}$$

$$[T_{\mu\nu}, K_{\alpha\beta}]_{q_2^{-2}} = -\frac{H}{q_2}\left(\delta_{\beta\mu}K_{\alpha\nu} - \delta_{\alpha\mu}K_{\beta\nu}\right), \tag{18}$$

$$[T_{\mu\nu}, L_{\alpha\beta}]_{q_2^2} = \frac{H}{q_2}\left(\delta_{\nu\alpha}L_{\mu\beta} - \delta_{\nu\beta}L_{\mu\alpha}\right), \tag{19}$$

$$[K_{\mu\nu}, K_{\alpha\beta}] = [L_{\mu\nu}, L_{\alpha\beta}] = 0, \tag{20}$$

$$[K_{\mu\nu}, L_{\alpha\beta}]_{q_2^4} = q_2^2 H\left(\delta_{\mu\alpha}T_{\beta\nu} - \delta_{\nu\alpha}T_{\beta\mu} + \delta_{\nu\beta}T_{\alpha\mu} - \delta_{\mu\beta}T_{\alpha\nu}\right), \tag{21}$$

and

$$[T_{\mu\nu}, H] = 0, \tag{22}$$

$$[K_{\mu\nu}, H]_{q_2^2} = 0, \tag{23}$$

$$[L_{\mu\nu}, H]_{q_2^{-2}} = 0. \tag{24}$$

Equations (17)–(24) define a deformed Lie algebra $so_{q_2}(2m, R)$. It is noteworthy to say that (5) and (10) as well as (17) and (22) show that the deformed Lie algebras $sp_{q_1}(2n, R)$ and $so_{q_2}(2m, R)$ have an exact $u(n)$ and $u(m)$ undeformed Lie subalgebra, respectively. It should be pointed out that (2) and (14) give the construction proposed by Schwinger [16] in terms of the q-deformed Newton oscillators.

Furthermore, it is interesting to investigate whether q-deformed Jacobi identities can be satisfied solely by using the equations given in (2)–(12). The general q-deformed Jacobi identity was first introduced by [8] and applied to the Biedenharn–Macfarlane oscillator algebra [3]. In our case we use (5)–(12) to show that q_1-deformed Jabobi identities such as

$$\begin{aligned} &\left[K_{lp}, [L_{mn}, T_{ij}]_{q_1}\right]_{q_1} + q_1\left[L_{mn}, [T_{ij}, K_{lp}]_{q_1}\right]_{q_1^{-1}} \\ &\qquad + \left[T_{ij}, [K_{lp}, L_{mn}]_{q_1^2}\right] = 0, \\ &\left[K_{ij}, [K_{lm}, T_{pr}]_{q_1}\right]_{q_1} + q_1\left[K_{lm}, [T_{pr}, K_{ij}]_{q_1}\right]_{q_1^{-1}} \\ &\qquad + \left[T_{pr}, [K_{ij}, K_{lm}]_{q_1^2}\right] = 0, \\ &\left[L_{ij}, [L_{lm}, T_{pr}]_{q_1}\right]_{q_1} + q_1\left[L_{lm}, [T_{pr}, L_{ij}]_{q_1}\right]_{q_1^{-1}} \\ &\qquad + \left[T_{pr}, [L_{ij}, L_{lm}]_{q_1^2}\right] = 0, \end{aligned}$$

are satisfied.

Before going to the construction of the deformed superoscillator algebra mentioned, we need to say a few words on the realizations of undeformed and deformed Lie algebras in terms of oscillators. The importance of oscillator construction of groups appearing in supergravity theories was recognized two decades ago [17]. Furthermore, Schwinger's construction of the classical Lie algebras has been well known, and its possible relations to the classical Lie algebras constructed with other approaches have been investigated [18,19]. However, the q-deformed $su(2)$ construction by Biedenharn–Macfarlane [3], which is a generalization of Schwinger's construction of the Lie algebra $su(2)$, has been used in some problems of physical interest such as the Bloch electron problem [20]. Another interesting application of deformed Lie algebras via the Schwinger construction is to two-dimensional statistical lattice models. These models exploit the undeformed fermionic oscillators which are transmuted into anyonic oscillators [21]. A two-parameter extension of the deformed Lie algebras using the Schwinger approach has also been studied and the resulting two-parameter quantum algebra together with its representation is found in terms of p,q-deformed harmonic oscillators [22,23].

3 The construction of the deformed Lie superalgebra $osp_{q_1,q_2}(2n \mid 2m, R)$

We now investigate the deformed construction of the Lie superalgebra mentioned above by using the q-deformed bosonic and fermionic Newton oscillators. We specifically consider the commutation relations in (1) and (13) together with the relations

$$\begin{aligned} a_i f_\mu &= r f_\mu a_i, \\ a_i f_\mu^+ &= s f_\mu^+ a_i, \end{aligned} \tag{25}$$

where r and s are parameters which can be determined from the deformed anti-commutation relations as

$$\begin{aligned} s &= \sqrt{q_1 q_2}, \\ r &= \sqrt{\frac{q_1}{q_2}}, \end{aligned} \tag{26}$$

where q_1 and q_2 are again deformation parameters of the bosonic and the fermionic Newton oscillators, respectively. The deformed generators given in (2)–(4), (14)–(16), and the fermionic generators

$$\begin{aligned} T_{i\mu} &= a_i f_\mu^+, \quad T_{\nu j} = f_\nu a_j^+, \\ K_{i\mu} &= a_i f_\mu, \quad L_{i\mu} = a_i^+ f_\mu^+ \end{aligned} \tag{27}$$

satisfy the following commutation and anti-commutation relations:

$$[K_{\mu\nu}, T_{i\lambda}]_{q_2^3 q_1^{-1}} = q_2^2 H \left(\delta_{\nu\lambda} K_{i\mu} - \delta_{\mu\lambda} K_{i\nu}\right), \tag{28}$$

$$[K_{\mu\nu}, L_{i\lambda}]_{q_1 q_2^3} = \sqrt{\frac{q_2^3}{q_1}} H \left(\delta_{\nu\lambda} T_{\mu i} - \delta_{\mu\lambda} T_{\nu i}\right), \tag{29}$$

$$[L_{\mu\nu}, T_{\lambda j}]_{q_1 q_2^{-3}} = \sqrt{\frac{q_1}{q_2^5}} H \left(-\delta_{\lambda\mu} L_{j\nu} + \delta_{\lambda\nu} L_{j\mu}\right), \tag{30}$$

$$[L_{\mu\nu}, K_{i\lambda}]_{q_1^{-1} q_2^{-3}} = -\frac{H}{q_2^3} \left(\delta_{\lambda\mu} T_{i\nu} - \delta_{\lambda\nu} T_{i\mu}\right), \tag{31}$$

$$[K_{ij}, T_{\nu k}]_{q_1^3 q_2^{-1}} = \sqrt{q_1^3 q_2} H \left(\delta_{jk} K_{i\nu} + \delta_{ik} K_{j\nu}\right), \tag{32}$$

$$[K_{ij}, L_{k\mu}]_{q_1^3 q_2} = q_1 H \left(\delta_{jk} T_{i\mu} + \delta_{ik} T_{j\mu}\right), \tag{33}$$

$$[L_{ij}, T_{k\mu}]_{q_2 q_1^{-3}} = -\frac{H}{q_1^2} \left(\delta_{ki} L_{j\mu} + \delta_{kj} L_{i\mu}\right), \tag{34}$$

$$[L_{ij}, K_{k\mu}]_{q_2^{-1} q_1^{-3}} = -\frac{H}{\sqrt{q_1^5 q_2}} \left(\delta_{ki} T_{\mu j} + \delta_{kj} T_{\mu i}\right), \tag{35}$$

$$[T_{ij}, T_{k\mu}]_{q_2 q_1^{-1}} = -\frac{H}{q_1} \delta_{ki} T_{j\mu}, \tag{36}$$

$$[T_{ij}, T_{\nu k}]_{q_1 q_2^{-1}} = \frac{H}{q_1} \delta_{jk} T_{\nu i}, \tag{37}$$

$$[T_{ij}, K_{k\mu}]_{q_1^{-1} q_2^{-1}} = -\frac{H}{q_1} \delta_{ki} K_{j\mu}, \tag{38}$$

$$[T_{ij}, L_{k\mu}]_{q_1 q_2} = \frac{H}{q_1} \delta_{jk} L_{i\mu}, \tag{39}$$

$$[T_{\mu\nu}, T_{i\lambda}]_{q_2 q_1^{-1}} = \frac{H}{q_2} \delta_{\nu\lambda} T_{i\mu}, \tag{40}$$

$$[T_{\mu\nu}, T_{\lambda j}]_{q_1 q_2^{-1}} = -\frac{H}{q_2} \delta_{\lambda\mu} T_{\nu j}, \tag{41}$$

$$[T_{\mu\nu}, K_{i\lambda}]_{q_1^{-1} q_2^{-1}} = -\frac{H}{q_2} \delta_{\lambda\mu} K_{i\nu}, \tag{42}$$

$$[T_{\mu\nu}, L_{i\lambda}]_{q_1 q_2} = \frac{H}{q_2} \delta_{\nu\lambda} L_{i\mu}, \tag{43}$$

$$[T_{ij}, K_{\mu\nu}]_{q_2^{-2}} = [T_{ij}, L_{\mu\nu}]_{q_2^2} = 0, \tag{44}$$

$$[K_{ij}, T_{\mu\nu}]_{q_1^2} = [L_{ij}, T_{\mu\nu}]_{q_1^{-2}} = 0, \tag{45}$$

$$\begin{aligned} [T_{ij}, T_{\mu\nu}] &= [K_{ij}, K_{\mu\nu}]_{q_1^2 q_2^{-2}} = [K_{ij}, L_{\mu\nu}]_{q_1^2 q_2^2} \\ &= 0, \\ [K_{ij}, T_{k\mu}]_{q_1 q_2} &= [K_{ij}, K_{k\mu}]_{q_1 q_2^{-1}} = [L_{ij}, K_{\mu\nu}]_{q_1^{-2} q_2^{-2}} \\ &= 0, \\ [L_{ij}, L_{\mu\nu}]_{q_2^2 q_1^{-2}} &= [L_{ij}, T_{\nu k}]_{q_1^{-1} q_2^{-1}} = [L_{ij}, L_{k\mu}]_{q_2 q_1^{-1}} \\ &= 0, \\ [K_{\mu\nu}, T_{\lambda j}]_{q_1 q_2} &= [K_{\mu\nu}, K_{i\lambda}]_{q_2 q_1^{-1}} = [L_{\mu\nu}, T_{i\lambda}]_{q_1^{-1} q_2^{-1}} \\ &= 0, \\ [L_{\mu\nu}, L_{i\lambda}]_{q_1 q_2^{-1}} &= 0, \end{aligned} \tag{46}$$

$$\{T_{i\mu}, T_{\nu j}\}_{q_1^2 q_2^{-2}} = q_1 H \left(\frac{q_1}{q_2} \delta_{\nu\mu} T_{ji} + \delta_{ij} T_{\mu\nu}\right), \tag{47}$$

$$\{T_{i\mu}, K_{j\nu}\}_{q_2^{-2}} = \sqrt{\frac{q_1^3}{q_2^3}} H \delta_{\nu\mu} K_{ij}, \tag{48}$$

$$\{T_{i\mu}, L_{j\nu}\}_{q_1^2} = \sqrt{\frac{q_1}{q_2}} H \delta_{ij} L_{\mu\nu}, \tag{49}$$

$$\{T_{\nu j}, K_{i\mu}\}_{q_1^{-2}} = \frac{q_2}{q_1} H \delta_{ij} K_{\mu\nu}, \tag{50}$$

$$\{T_{\nu j}, L_{i\mu}\}_{q_2^2} = \frac{q_2}{q_1} H \delta_{\nu\mu} L_{ij}, \tag{51}$$

$$\{K_{i\mu}, L_{j\nu}\}_{q_1^2 q_2^2} = \sqrt{q_1 q_2} H \left(q_1 \delta_{\mu\nu} T_{ji} - q_2 \delta_{ij} T_{\nu\mu}\right), \tag{52}$$

$$\begin{aligned} \{T_{i\mu}, T_{j\nu}\} &= \{T_{\nu j}, T_{\mu i}\} = \{K_{i\mu}, K_{j\nu}\} \\ &= \{L_{i\mu}, L_{j\nu}\} = 0, \end{aligned} \tag{53}$$

and also,

$$[T_{i\mu}, H]_{q_1 q_2^{-1}} = 0, \tag{54}$$

$$[T_{\nu j}, H]_{q_2 q_1^{-1}} = 0, \tag{55}$$

$$[K_{i\mu}, H]_{q_1 q_2} = 0, \tag{56}$$

$$[L_{i\mu}, H]_{q_1^{-1} q_2^{-1}} = 0, \tag{57}$$

where i, j, k, l, m, n are bosonic indices and $\mu, \nu, \alpha, \beta, \lambda$ are fermionic indices. These commutation relations together with (5)–(9) and (17)–(21) complete the construction of the deformed Lie superalgebra $osp_{q_1,q_2}(2n \mid 2m, R)$. It should be mentioned that the generators T_{ij}, K_{ij}, L_{ij} and $T_{\mu\nu}, K_{\mu\nu}, L_{\mu\nu}$ generate the even part of this deformed Lie superalgebra as the deformed Lie subalgebra $sp_{q_1}(2n, R) \times so_{q_2}(2m)$ where $sp_{q_1}(2n, R)$ is non-compact with the maximal compact subalgebra $u(n)$,, and $so_{q_2}(2m)$ is compact and contains a $u(m)$ subalgebra. Furthermore, the generators $T_{i\mu}$ and $T_{\nu j}$ generate the Lie superalgebra $u(n \mid m)$ under the Lie superproduct which is defined as the anti-commutator between the two fermionic generators given in (47). The standard construction of the Lie superalgebra $osp(2n \mid 2m, R)$ can be recovered from $osp_{q_1,q_2}(2n \mid 2m, R)$ by applying the limits $q_1 \to 1$ and $q_2 \to 1$ [24,10].

We now wish to review the $U(n \mid m)$ invariance of the q-deformed supersymmetric Newton oscillators for which we use the even and the odd generators of the superalgebra constructed. The unitary supergroup action in the super Fock space can be represented by the operator

$$u\,(a, f) = \exp\left(\mathrm{i}\omega^a T_a + \mathrm{i}\theta^f T_f\right) \tag{58}$$

where T_a denotes the even generators $T_{ij}, T_{\mu\nu}$ given by (2) and (14), and T_f denotes the odd generators $T_{i\mu}$, $T_{\nu j}$ given by (27). ω^a and θ^f are real bosonic and fermionic parameters, respectively. Thus, the invariance mentioned follows directly from

$$\tilde{a}_i \to u_{ik} a_k, \quad u_{ik} u^{+}_{jk} = \delta_{ij} \tag{59}$$

where u is defined by means of (58). One can verify choosing (1) or (13) that q-deformed supersymmetric Newton oscillators are invariant under the undeformed Lie supergroup $U(n \mid m)$.

4 Discussion

As a matter of fact, the deformed supersymmetric oscillators are a new research trend in theoretical physics. Earlier q-deformed versions of supersymmetric algebras have been proposed by Parthasarathy [25] and Spiridonov [26]. In connection with this, an interesting study related to the q-deformed $N = 2$ supersymmetric algebra has been published in [27]. The number operator and the Fock space representation of a q-deformed supersymmetric oscillator is suggested by [28]. The remarkable point is that the investigations mentioned have combined both conventional fermion and boson operators as well as their q-deformed versions. Within the context of a more fundamental theory such as supersymmetric quantum mechanics (SSQM) [29], the parasupersymmetric investigations including parabosons and parafermions have also attracted a lot of attention in connection with parastatistics developments such as the q-deformed parabosonic systems using the Green's ansatz by the Macfarlane method [30]. In particular, some questions related to "how SSQM can be deformed by means of these developments" are other important points and the realizations of the orthosymplectic Lie superalgebras together with their representations are intimately related to this deformed SSQM [31].

Furthermore, we wish to remark that the Hopf algebra structure of the deformed Lie superalgebra $osp_{q_1,q_2}(2n \mid 2m, R)$ in terms of the q-deformed supersymmetric Newton oscillators is an open problem, together with its possible relations to the other deformed versions of the Lie superalgebra $osp(2n \mid 2m, R)$.

As a final remark, we have constructed a two-parameter deformed Lie superalgebra $osp_{q_1,q_2}(2n \mid 2m, R)$ by using the q-deformed supersymmetric Newton oscillators invariant under the undeformed Lie supergroup $U(n \mid m)$. Due to the growing interest in applications of supersymmetry, we hope and conjecture that the realization of the Lie superalgebra $osp_{q_1,q_2}(2n \mid 2m, R)$ will be a powerful tool in such investigations.

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